

# Comment on “Integrability of the Rabi Model”

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(Dated: October 6, 2015)

PACS numbers: 03.65.Ge, 02.30.Ik, 42.50.Pq

In a recent Letter [1], Braak proved that the spectrum of the Rabi model  $H = \omega a^\dagger a + g\sigma_z(a + a^\dagger) + \Delta\sigma_x$  consists of regular and exceptional spectrum. The *necessary and sufficient* condition for the occurrence of the exceptional eigenvalues  $E_n = n\omega - g^2/\omega$  reads  $K_n(n\omega) = 0$ , which is just the condition for Judd’s solutions [2]. In this Comment, we show that  $K_n(n\omega) = 0$  is only a *sufficient* but not *necessary* condition for the occurrence of  $E_n$ . In other words, the set of Judd’s solutions is just a subset of the exceptional eigenvalues.

We first solve the spectrum using Hill’s determinant method [3]. In Bargmann representation, the eigenvalue equation reads ( $\hbar = \omega = 1$ )

$$[z\partial_z + g\sigma_z(z + \partial_z) + \Delta\sigma_x]\psi(z) = E\psi(z), \quad (1)$$

where  $\psi(z)$  is an entire function of  $z$ . By writing  $\psi(z) = e^{-gz} \sum_{m=0}^{\infty} (z+g)^m (p_m, q_m)^T$ , inserting into Eq.(1) and eliminating  $p_m$ , we obtain a set of linear equations for  $q_m$ ,  $W_0^\infty q_0^\infty = 0$ , where  $q_0^\infty = (q_0, q_1, \dots, q_m)^T$  and the coefficient matrix

$$W_0^m = \begin{bmatrix} a_0 & -b_0 & 0 & \cdots & \cdots \\ -c_1 & a_1 & -b_1 & \cdots & \cdots \\ 0 & -c_2 & a_2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & -c_m & a_m \end{bmatrix}$$

is tridiagonal, with  $a_m = (m-x)(m-x+4g^2) - \Delta^2$ ,  $b_m = 2g(m+1)(m-x)$ ,  $c_m = 2g(m-x)$ , and  $x = E + g^2$ . Following the analysis in Ref. [3], the eigenvalues can be determined by the equation

$$\tilde{D}(x) \equiv \lim_{m \rightarrow \infty} \frac{\Gamma^2(m+1+x)}{\Gamma^4(m+1)} \det[W_0^m] = 0, \quad (2)$$

and the corresponding coefficients  $q_0^\infty$  is the minimal solution of  $W_0^\infty q_0^\infty = 0$ .

Now consider the eigenvalues of the form  $x_n = n$ , with nonnegative integer  $n$ . Since  $a_n = -\Delta^2$ ,  $b_n = c_n = 0$ , we have

$$\tilde{D}(x_n) = -\Delta^2 \det[W_0^{n-1}] \lim_{m \rightarrow \infty} \frac{[(n+m)!]^2}{(m!)^4} \det[W_{n+1}^m],$$

with  $\det[W_0^{-1}] \equiv 1$ . The eigenvalue equation (2) then leads to (i)  $\Delta = 0$ , or (ii)  $\det[W_0^{n-1}] = 0$ , or (iii)

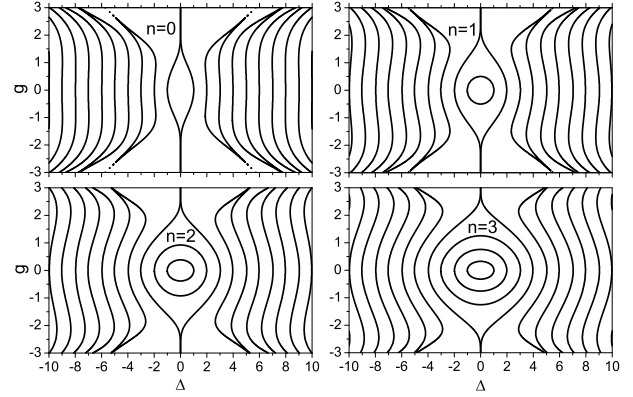


FIG. 1. Solutions of  $\lim_{m \rightarrow \infty} \frac{[(n+m)!]^2}{(m!)^4} \det[W_{n+1}^m] = 0$  for  $n = 0, 1, 2, 3$  in the  $\Delta$ - $g$  plane.

$\lim_{m \rightarrow \infty} \frac{[(n+m)!]^2}{(m!)^4} \det[W_{n+1}^m] = 0$ . The case (i) is the adiabatic limit and exactly solvable. The case (ii) corresponds to the isolated exact solutions and the condition  $\det[W_0^{n-1}] = 0$  is equivalent to  $K_n(n\omega) = 0$  in Braak’s paper [1]. The corresponding eigenvector  $q_0^\infty$  has the form  $(q_0, q_1, \dots, q_{n-1}, 0, 0, \dots)^T$ . The case (iii) is not discussed in Braak’s paper and is the main point of this Comment. Some numerical solutions in this case are plotted in Fig.1. We see that the solutions for a given integer  $n$  contain two parts: (α)  $n$  closed loops around the center on which  $\det[W_0^{n-1}] = 0$  is also valid due to the double degeneracy of the corresponding level  $x_n$ , and (β) infinitely many lines passing through the points  $g = 0, \Delta = \pm(n+1), \pm(n+2), \dots$ , on which the level  $x_n$  is not degenerate. These lines are neglected in Braak’s solution. The corresponding eigenvector  $q_0^\infty$  in case (iii) has the form  $(0, 0, \dots, 0, q_{n+1}, q_{n+2}, \dots)^T$ , i.e., the first  $n+1$  components are zero.

In conclusion, using Hill’s determinant method [3] we have shown that the set of Judd’s solutions is only a subset of all the eigenvalues with the form  $E_n = n\omega - g^2/\omega$  in the spectrum of the Rabi model. Therefore Braak’s solution is not complete. We note that this structure of the exceptional spectrum has also been derived recently in Ref.[4], but their method is quite different from ours.

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